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# Electromagnetic potentials basis for energy density and power flux

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## Abstract

In rounding out the education of students in advanced courses in applied electromagnetics it is incumbent on us as mentors to raise issues that encourage appreciation of certain subtle aspects that are often overlooked during first exposure to the field. One of these has to do with the interplay between fields and potentials, with the latter often seen as just a convenient mathematical artifice useful in solving Maxwell's equations. Nonetheless, to those practiced in application it is well understood that various alternatives in the use of fields and potentials are available within electromagnetic (EM) theory for the definitions of energy density, momentum transfer, EM stress–energy tensor, and so forth. Although the various options are all compatible with the basic equations of electrodynamics (e.g., Maxwell's equations, Lorentz force law, gauge invariance), nonetheless certain alternative formulations lend themselves to being seen as preferable to others with regard to the transparency of their application to physical problems of interest. Here we argue for the transparency of an energy density/power flux option based on the EM potentials alone.

Keywords: electromagnetic potentials, vector potential, scalar potential, EM energy density, EM power flux



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## 1. Introduction

The standard definition encountered in textbooks (and in mainstream use) for energy density  $u$  and power flux  $\mathbf{S}$  in EM fields is given by

$$u(\mathbf{r}, t) = \frac{1}{2}[\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2], \quad (1a)$$

$$\mathbf{S}(\mathbf{r}, t) = (\mathbf{E} \times \mathbf{H}). \quad (1b)$$

These definitions of energy density and power flux are the ones encountered by students in their first exposure to the topic. However, one can argue that this formulation, even though resulting in paradoxes, owes its staying power more to historical development than to transparency of application in many cases. One oft-noted paradox in the literature, for example, is the (mathematical) apparency, though unobservable, of momentum transfer at a given location in static superposed electric and magnetic fields, seemingly implied by equation (1b) above<sup>1</sup>. A second example is highlighted in Feynman's commentary that use of the standard  $(\mathbf{E} \times \mathbf{H})$  Poynting vector approach leads to '... a peculiar thing: when we are slowly charging a capacitor, the energy is not coming down the wires; it is coming in through the edges of the gap', a seemingly absurd result in his opinion, regardless of its uniform acceptance as a correct description [1].

As developed in detail here, definitions for energy transfer based on the use of EM potentials alone is shown to yield results more in keeping with intuition for conundrums such as the above, and in the process provide additional insight into the significance of the potentials (even in classical theory) that extend beyond simply that of constituting a placeholder for EM fields.

## 2. Background

Though little discussed, there exists a rich literature on alternative definitions of EM energy density and power flux, the general purpose of which has been to attempt to bring clarity to some of the ambiguities associated with the standard definitions. In an early paper by Slepian, for example, no less than eight alternatives to the standard definitions are considered, all congruent with Maxwell's equations [2]. That is, once integrated over a volume of interest the net energy and power flow resulting from use of the various alternative expressions are all in agreement; only the density *distributions* and their associated interpretations differ.

As an example, for Maxwell fields the standard expression involving energy density and power flux

$$\frac{\partial}{\partial t}u + \nabla \cdot \mathbf{S} = \frac{\partial}{\partial t} \frac{1}{2}(\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) \quad (2)$$

is trivially found by the use of vector identities to be equally satisfied by

$$u = \frac{1}{2}(\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2) - \nabla \cdot \mathbf{Y}, \quad (3a)$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} + \nabla \times \mathbf{X} + \frac{\partial}{\partial t}\mathbf{Y}, \quad (3b)$$

where  $\mathbf{X}$  and  $\mathbf{Y}$  are arbitrary vector fields [3].

<sup>1</sup> This has on occasion led to time-consuming debates in the engineering literature as to the feasibility of certain forms of electromagnetic propulsion.

Application of the above option for the case  $\mathbf{X} = \varphi\mathbf{H}$ ,  $\mathbf{Y} = \mathbf{0}$ , yields [4]

$$\mathbf{S} = -\frac{\partial\mathbf{A}}{\partial t} \times \mathbf{H} + \varphi\left(\mathbf{j} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right), \quad (4)$$

where  $(\mathbf{A}, \varphi)$  are the vector and scalar potentials defined by

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (5)$$

From equation (4) we find that this alternative form for the power flux  $\mathbf{S}$  addresses, and provides alternative interpretations for, the conundrums referred to in the opening paragraph above in that (a) by virtue of the time derivatives in the definition of  $\mathbf{S}$  no energy flow is attributed to static superposed electric and magnetic fields in free space, and (b) given that  $\mathbf{S} \propto \varphi\mathbf{j}$ , detailed calculation (see section 4.2) shows that energy is revealed as flowing down the wire during the charging of a capacitor. Therefore, at least for this pair of problems, the definition of  $\mathbf{S}$  given by equation (4) appears more aligned with our intuition than equation (1b) relative to the physical processes involved.

Unfortunately, in the absence of a well-structured theoretical guideline, selection of the above and similar modifications, typically involving mixed field and potential terms, remains an ad hoc procedure at best. In the next section we consider a universally applicable, systematic treatment based on the use solely of EM potentials alone.

### 3. Potentials-based definitions

Although use of the vector and scalar potentials  $(\mathbf{A}, \varphi)$  in place of the EM fields  $(\mathbf{E}, \mathbf{B})$  is considered simply to be an option in classical theory, in quantum theory they are understood to be more fundamental than the derivative electric and magnetic fields  $(\mathbf{E}, \mathbf{B})$  which are the ‘coin of the realm’ in ordinary classical theory. In classical electrodynamics the choice of which variable pair to use is arbitrary, and the overall resulting predictions in terms of observables are indistinguishable. Nonetheless, cogent arguments can be made (and is made here) that the  $(\mathbf{A}, \varphi)$  approach is to be preferred, even in classical EM theory, because of increased transparency in application.

By virtue of the freedom in EM theory to choose a gauge (gauge invariance), when employing the  $(\mathbf{A}, \varphi)$  potentials it is convenient to choose the Lorenz gauge<sup>2</sup>

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial\varphi}{\partial t}. \quad (6)$$

This results in simplified wave equations fully equivalent to Maxwell’s wave equations for  $\mathbf{E}$  and  $\mathbf{B}$

$$\nabla^2\varphi - \frac{1}{c^2} \frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}, \quad \nabla^2\mathbf{A} - \frac{1}{c^2} \frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}, \quad (7)$$

in which the scalar potential  $\varphi$  is determined by the charge density  $\rho$  alone, and the vector potential  $\mathbf{A}$  is determined by the current density  $\mathbf{j}$  alone. Key to the development here, it is the dependence on separate source terms for the two potentials that contributes to an independence that leads to transparency in application, as will be seen below. The solutions to equation (7) are given by the retarded Green’s functions

<sup>2</sup> We note in passing that should evidence ever be forthcoming for a finite photon mass, regardless of how infinitesimal, then gauge invariance would be broken and the Lorenz gauge would be required. See, e.g., [5].

$$\varphi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} dV', \quad \mathbf{A}(\mathbf{r}, t) = \int \frac{\mu_0 \mathbf{j}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi |\mathbf{r} - \mathbf{r}'|} dV'. \quad (8)$$

Given a coordinate system,  $\mathbf{r}$  denotes the location relative to the origin where potentials are measured due to potential sources located at  $\mathbf{r}'$ . This convention is used throughout the paper.

By use of equations (6)–(8) and the definitions provided in equation (5), the usual Maxwell equations in terms of  $\mathbf{E}$  and  $\mathbf{B}$  as driven by charge and current densities can be derived.

It is at this juncture that the approach presented here differs substantially from the usual approach concerning the definitions of EM energy density and power flux, and that as a consequence can provide for increased transparency in application. First, we note that the scalar potential  $\varphi$  and vector potential  $\mathbf{A}$  each satisfy independent wave equations. For this case involving unmixed-variable wave equations, in the theory of partial differential equations there arise, specifically in terms of gradients, generalised natural measures of associated ‘energy density’ and ‘power flux’ (and associated conservation laws) for the variables of interest (here  $\varphi$  and  $\mathbf{A}$ ) [6]. Secondly, we note from equation (7) that the field effects from charges and currents can be seen as resulting from two *noninteracting* physical processes described, respectively, by the scalar and vector potentials  $\varphi$  and  $\mathbf{A}$ . We can therefore anticipate that reasonable and satisfactory definitions for the densities of momenta and energy should be expressible in terms of *unmixed* derivatives of  $\varphi$  and  $\mathbf{A}$ . Though little referenced, such approaches have been advanced in the literature from time to time in various forms and constitute the basis for the present investigation [7, 8].

In place of the standard definition for EM energy density, equation (1a), we take as our definition

$$u(\mathbf{r}, t) = u_A - u_\varphi + \rho\varphi, \quad (9)$$

where  $u_A$  is an energy density defined in terms of derivatives of the vector potential only<sup>3</sup>

$$u_A(\mathbf{r}, t) = \frac{1}{2\mu_0} \sum_i \left[ \frac{1}{c^2} \left( \frac{\partial A_i}{\partial t} \right)^2 + |\nabla A_i|^2 \right], \quad (10)$$

and  $u_\varphi$  is an energy density defined in terms of derivatives of the scalar potential only

$$u_\varphi(\mathbf{r}, t) = \frac{1}{2}\epsilon_0 \left[ \frac{1}{c^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 + |\nabla \varphi|^2 \right]. \quad (11)$$

In place of the standard definition for EM power flux equation (1b), we take as our definition

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{S}_A - \mathbf{S}_\varphi + \varphi \mathbf{j}, \quad (12)$$

with  $\mathbf{S}_A$  and  $\mathbf{S}_\varphi$  defined in terms of their respective (and separate) derivatives of the potentials as well

$$\mathbf{S}_A(\mathbf{r}, t) = -\frac{1}{\mu_0} \sum_i \left( \frac{\partial A_i}{\partial t} \right) \nabla A_i, \quad (13)$$

<sup>3</sup> For simplicity of notation it is to be understood here that in the application of equations (10) (and (13) following) the gradient operator  $\nabla = \hat{\mathbf{i}}_x \frac{\partial}{\partial x} + \hat{\mathbf{i}}_y \frac{\partial}{\partial y} + \hat{\mathbf{i}}_z \frac{\partial}{\partial z}$  and vector components  $A_i$  are to be expressed in terms of rectangular coordinates  $i = x, y, z$ .

$$\mathbf{S}_\varphi(\mathbf{r}, t) = -\varepsilon_0 \left( \frac{\partial \varphi}{\partial t} \right) \nabla \varphi. \quad (14)$$

The associated Lorentz power density associated with charged-particle motion is given by an expression that parallels that based on densities defined in terms of the electric and magnetic fields ( $\mathbf{E}$ ,  $\mathbf{B}$ )

$$p_L = -\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{S}. \quad (15)$$

Finally, it can be shown that the wave equation (7) are the associated Euler–Lagrange variational equations that can be derived from a Lagrangian density that is both relativistically covariant and *has no cross-interaction terms* between the scalar and vector potential terms  $\varphi$  and  $\mathbf{A}$ :

$$\mathcal{L} = -\frac{1}{2\mu_0} (\partial^\alpha A_\beta)(\partial_\alpha A^\beta) - j^\alpha A_\alpha = \mathcal{L}_A(\mathbf{r}, t) - \mathcal{L}_\varphi(\mathbf{r}, t) + \mathbf{j} \cdot \mathbf{A} - \rho\varphi, \quad (16)$$

where

$$\mathcal{L}_A(\mathbf{r}, t) = \frac{1}{2\mu_0} \sum_i \left[ \frac{1}{c^2} \left( \frac{\partial A_i}{\partial t} \right)^2 - |\nabla A_i|^2 \right] \quad (17)$$

and

$$\mathcal{L}_\varphi(\mathbf{r}, t) = \frac{1}{2} \varepsilon_0 \left[ \frac{1}{c^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 - |\nabla \varphi|^2 \right]. \quad (18)$$

## 4. Applications

### 4.1. Radiation fields

For our first example we examine the case of fields radiated from a short dipole antenna (length  $l$  small compared with the wavelength of the radiated signal) that carries a current  $i = I \sin \omega t$ . In this specific case, unlike those that follow in later sections, both the energy density and power flux distributions are found to be of the same form whether evaluated using the standard EM-fields approach or the potentials-based approach being examined in detail here.

For a dipole located at  $\mathbf{r}' = 0$ , oriented vertically in the  $x$  direction, the vector potential at a distance  $r \gg l$  between dipole and field point is determined from equation (8) as

$$A_x = \frac{1}{4\pi} \int_{-l/2}^{l/2} \frac{\mu_0 I \sin(\omega t - kr)}{r} dx \approx \frac{\mu_0 I l}{4\pi r} \sin(\omega t - kr), \quad (19)$$

where  $k = \omega/c$ , and for dipole length  $l$  small compared to wavelength and  $r \gg l$ , the numerator and denominator of the integrand, respectively, are each nearly constant. In spherical coordinates ( $A_r = A_x \cos \theta$ ,  $A_\theta = -A_x \sin \theta$ ,  $A_\phi = 0$ ) we then have

$$A_r = \frac{\mu_0 I l}{4\pi r} \cos \theta \sin(\omega t - kr), \quad A_\theta = -\frac{\mu_0 I l}{4\pi r} \sin \theta \sin(\omega t - kr), \quad A_\phi = 0, \quad (20)$$

where  $\theta$  denotes the angle measured from the  $x$  axis of the dipole orientation to that of the  $r$ -directed vector. The corresponding far-field scalar potential is found from equation (6) to be

$$\varphi = \frac{Il \cos \theta}{4\pi r} \sqrt{\frac{\mu_0}{\varepsilon_0}} \sin(\omega t - kr). \quad (21)$$

The derivative electric and magnetic fields are then found by equation (5) to be

$$E_r = E_\phi = 0, \quad E_\theta = \frac{\mu_0 Il \omega}{4\pi r} \sin \theta \cos(\omega t - kr), \quad (22)$$

$$B_r = B_\theta = 0, \quad B_\phi = \frac{\mu_0 Il k}{4\pi r} \sin \theta \cos(\omega t - kr), \quad (23)$$

where for the far field only terms in  $1/r$  are kept.

To determine the energy density  $u$  and power flux  $\mathbf{S}$  distributions, we apply equation (1) for the standard EM-field-based approach, and equations (9)–(14) for the potentials-based approach. (Recall from footnote 3 that for the latter, the gradient operator  $\nabla$  and vector components  $A_i$  are to be expressed in rectangular coordinates.) The result is that for both the standard EM and the potentials-based approaches the outcome is

$$u(\mathbf{E}, \mathbf{B}) = u(\mathbf{A}, \varphi) = \mu_0 \left[ \frac{Ilk \sin \theta \cos(\omega t - kr)}{4\pi r} \right]^2, \quad (24)$$

$$\mathbf{S}(\mathbf{E}, \mathbf{B}) = \mathbf{S}(\mathbf{A}, \varphi) = \mathbf{1}_r \sqrt{\frac{\mu_0}{\varepsilon_0}} \left[ \frac{Ilk \sin \theta \cos(\omega t - kr)}{4\pi r} \right]^2. \quad (25)$$

Therefore, in this (as it turns out, atypical) case, both the energy density and power flux distributions are found to be of the same form whether evaluated on the basis of the standard EM-fields approach or the potentials-based approach.

#### 4.2. Superposed static electric and magnetic fields

For the second example we examine a configuration in which the difference between the standard EM-fields approach and the potentials-based approach is significant with regard to interpretations. Specifically, we re-address the issue of energy transfer for the case of superposed static electric and magnetic fields initially considered in section 2 above, here using an alternate route to the outcome predicted by equation (4). Equations (12)–(14) show that with the present potentials-based approach the power flux  $\mathbf{S}$  (and associated momentum transfer) depend on products involving time derivatives  $(\partial A_i / \partial t)$  and  $(\partial \varphi / \partial t)$  and therefore, again, do not attribute momentum transfer to superposed static field distributions. This is in contrast to the definition of momentum transfer in the standard formulation where power flux (at a point) is defined in terms of a crossed-field Poynting vector product  $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ . As alluded to earlier, the Poynting vector definition leads to a possible (mistaken) inference that momentum transfer accompanying power flux can be associated with crossed static electric and magnetic fields, even though there are no observable consequences of such (and, worse, the drawing of faulty conclusions that such momentum transfer can lead to, say, propulsive mechanisms). Though once fully integrated over boundary surfaces the two approaches, the potentials-based approach and the fields-based approach, lead to identical results, it is the point-by-point *distributions* that differ, with the  $(\mathbf{A}, \varphi)$  approach being more in harmony with our intuition concerning the relationship between causal charge/current sources and field effects.

#### 4.3. Power dissipation in a conductor

As a third example we consider a segment of a current-carrying wire of radius  $R$  and length  $L$  oriented in, say, the  $z$  direction in which a steady constant-current-density  $\mathbf{j} = \sigma \mathbf{E}$  flows, also in the  $z$  direction, where  $\sigma$  is the conductivity. The current-encircling azimuthal magnetic field is found from  $\nabla \times \mathbf{H} = \mathbf{j} \Rightarrow \int \mathbf{H} \cdot d\mathbf{l} = I$  to be  $\mathbf{H} = \hat{\mathbf{i}}_\theta |\mathbf{j}| R/2$ . By use of the standard Poynting flux expression equation (1b) this leads to an *inwardly directed* radial power flux and associated net power dissipation

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = -\hat{\mathbf{i}}_r \frac{|\mathbf{j}|^2 R}{2\sigma}, \quad (26a)$$

$$P = \int \mathbf{S} \cdot d\mathbf{a} = -\frac{\pi |\mathbf{j}|^2 R^2 L}{\sigma}. \quad (26b)$$

Application of the potential-based flux vector given by equations (12)–(14), on the other hand, gives a *z-directed* power flux aligned with the current flow, viz.

$$\mathbf{S} = \varphi \mathbf{j} \Rightarrow \mathbf{S}_u = \varphi_u \mathbf{j}, \quad \mathbf{S}_l = \varphi_l \mathbf{j}, \quad \mathbf{E} = \hat{\mathbf{i}}_z \frac{\varphi_l - \varphi_u}{L} = \frac{\mathbf{j}}{\sigma}, \quad (27)$$

where subscripts  $u$  and  $l$  refer to upper and lower parts of the wire segment (i.e.,  $+/-z$  locations relative to a chosen origin), and the  $z$ -directed electric field is expressed in terms of the potential gradient. From the above we calculate the *longitudinally directed* power flux and associated net power dissipation as

$$P = \int \mathbf{S} \cdot d\mathbf{a} = (\varphi_u - \varphi_l) |\mathbf{j}| \pi R^2 = -|\mathbf{E}| L |\mathbf{j}| \pi R^2 = -\frac{\pi |\mathbf{j}|^2 R^2 L}{\sigma}. \quad (28)$$

Though leading to the same result as equation (26b) with regard to net power dissipation, between the two options the potentials-based expression for the power flux *distribution* (being aligned with and delivered by the current rather than being inwardly field-directed) is, again, more in alignment with our natural intuition.

Let us now enquire as to the form of the energy density distributions for this example under the two approaches. (It turns out that, unlike the power flux distributions, they are identical—not always the case as will be seen in the example of section 4.4).

The energy density distribution for a current-carrying wire can be expressed in terms of either the magnetic field  $\mathbf{B}$  or the vector potential  $\mathbf{A}$  in accordance with the definitions provided in equations (1a) and (10), respectively. The two alternatives are given by:

$$u_B(\mathbf{r}, t) = \frac{1}{2} \mu_0 |\mathbf{H}|^2 = \frac{1}{2\mu_0} |\mathbf{B}|^2, \quad (29a)$$

$$u_A(\mathbf{r}, t) = \frac{1}{2\mu_0} \sum_i |\nabla A_i|^2. \quad (29b)$$

where, again, it is understood that  $|\nabla A_i|^2$  is to be expressed in Cartesian coordinates  $(x, y, z)$  for calculational purposes.

In a constant-current-density  $\mathbf{j}$ , current-carrying wire oriented in the  $z$  direction, the  $\mathbf{A}$  and  $\mathbf{B}$  field components in cylindrical coordinates, related by  $\mathbf{B} = \nabla \times \mathbf{A}$ , are given by [9]



$$B_r = B_z = 0, \quad B_\theta = \frac{\mu_0 |\mathbf{j}| r}{2}, \quad A_r = A_\theta = 0, \quad A_z = -\frac{\mu_0 |\mathbf{j}| r^2}{4} + C_1, \quad r \leq R \quad (30a)$$

$$B_r = B_z = 0, \quad B_\theta = \frac{\mu_0 |\mathbf{j}| R^2}{2r}, \quad A_r = A_\theta = 0, \quad A_z = -\frac{\mu_0 |\mathbf{j}| R^2}{2} \ln r + C_2, \quad r > R. \quad (30b)$$

In terms of either the magnetic field  $\mathbf{B}$  or vector potential  $\mathbf{A}$  given by equation (30), the energy density distributions equation (29) inside the current-carrying wire are identical for the fields-based and potentials-based approaches and are given by

$$u_B = u_A = \frac{\mu_0 |\mathbf{j}|^2 r^2}{8}, \quad r \leq R \quad (31a)$$

Outside, the  $\mathbf{B}$ - and  $\mathbf{A}$ -based energy density distributions are similarly identical, and are given by

$$u_B = u_A = \frac{\mu_0 |\mathbf{j}|^2 R^4}{8r^2}, \quad r \geq R \quad (31b)$$

Thus for the case of a current-carrying wire the energy density distributions  $u$  for the two field variables  $\mathbf{B}$  and  $\mathbf{A}$ , unlike the associated power flux distributions  $\mathbf{S}$ , are identical in their distributions, both inside and outside the wire. This is not the case for the following fourth example where the energy density distributions in the two approaches differ substantially.

#### 4.4. Magnetic solenoid

Here we consider a (near-infinite-length) cylindrical magnetic solenoid of radius  $R$ , containing a uniform-density magnetic field  $\mathbf{B}$  oriented in the  $z$  direction. The  $\mathbf{A}$  and  $\mathbf{B}$  field components in cylindrical coordinates, related by  $\mathbf{B} = \nabla \times \mathbf{A}$ , are determined from  $\int \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a}$ , with  $r$  being the distance from the central axis of the solenoid to the point of measurement:

$$B_r = B_\theta = 0, \quad B_z = B, \quad A_r = A_z = 0, \quad A_\theta = \frac{|\mathbf{B}| r}{2}, \quad r \leq R \quad (32a)$$

$$B_r = B_\theta = B_z = 0, \quad A_r = A_z = 0, \quad A_\theta = \frac{|\mathbf{B}| R^2}{2r}, \quad r \geq R \quad (32b)$$

In terms of the magnetic field  $\mathbf{B}$ , the energy densities given by equation (29) inside and outside the solenoid are given by

$$u_B = \frac{|\mathbf{B}|^2}{2\mu_0} \quad r \leq R, \quad (33a)$$

$$u_B = 0 \quad r > R. \quad (33b)$$

with associated energies  $E_B = \int u_B dV$  per unit length  $L$  given by

$$\frac{E_B}{L} = \frac{\pi R^2 |\mathbf{B}|^2}{2\mu_0} \quad r \leq R, \quad (34a)$$

$$\frac{E_B}{L} = 0 \quad r > R. \quad (34b)$$

That is, all the magnetostatic energy is confined *within* the solenoid in the fields-based approach.

In terms of the vector potential  $\mathbf{A}$  associated with the magnetic field  $\mathbf{B}$  the energy densities given by equation (10),  $u_A = \frac{1}{2\mu_0} \sum_i |\nabla A_i|^2$ , inside and outside the solenoid are given by

$$u_A^i = \frac{|\mathbf{B}|^2}{4\mu_0}, \quad (35a)$$

$$u_A^o = \frac{R^4 |\mathbf{B}|^2}{4\mu_0 r^4}, \quad (35b)$$

where in calculating the above we again express the vector potential in its Cartesian form before calculating the derivatives, i.e.

$$A_x = A_\theta(x, y) \sin \theta = A_\theta(x, y) \frac{y}{\sqrt{x^2 + y^2}}, \quad A_y = A_\theta(x, y) \cos \theta = A_\theta(x, y) \frac{x}{\sqrt{x^2 + y^2}}. \quad (36)$$

The associated energies  $E_A = \int u_A dV$  per unit length  $L$  are given by

$$\frac{E_A}{L} = \frac{\pi R^2 |\mathbf{B}|^2}{4\mu_0} \quad r \leq R, \quad (37a)$$

$$\frac{E_A}{L} = \frac{\pi R^2 |\mathbf{B}|^2}{4\mu_0} \quad r > R, \quad (37b)$$

i.e., are the same inside and outside the solenoid. Comparison of equation (37) with equation (34) shows that the net energies derived either in terms of the fields-based or potentials-based energy densities, though differently distributed, are equal, one being confined wholly within the solenoid ( $E_B$ ), the other ( $E_A$ ) being distributed 50% inside, 50% outside.

Given that all of the magnetic flux is confined to the interior of the solenoid, none outside, one infers (correctly) that there are no magnetic field effects to be detected by classical charge motion outside, given that the Lorentz force  $\mathbf{F} = q\mathbf{v} \times \mathbf{B} = 0$ . From a quantum viewpoint, however, despite the inability to detect classical effects on charges exterior to the solenoid, at the quantum level quantum interference effects of the vector potential  $\mathbf{A}$  exterior to the solenoid *can* be detected via  $\mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \int \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a}$  (Aharonov–Bohm, or  $A$ – $B$  effect). To the degree that one finds it intuitively appealing to consider that registration of a quantum effect exterior to the solenoid can be associated with changes in the vacuum's energy density surrounding the solenoid when energised, we note that in the potentials-based approach magnetostatic energy as defined by equation (37) does in fact reside not only in the region interior to the solenoid, but exterior to the solenoid as well. Though not absolutely required to be the case in the classical picture, this alternative, mathematically congruent finding dovetails in a reasonable way with registration of the  $A$ – $B$  effect.

## 5. Discussion

In the application of electromagnetic principles there has been a continuing development of various alternatives with regard to definitions involving the distributions of energy density and momentum transfer by EM fields defined in terms of the variables ( $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{A}$ ,  $\varphi$ ). This is a consequence of the fact that the distributions are not uniquely determined by Maxwell's

equations. Since in EM calculations all of the various (viable) options lead to identical predictions and outcomes with regard to net integrated energy density and power flux, from a mathematical viewpoint they are found to be equivalent as to net results. Therefore, strictly speaking, it is generally considered to be a matter of *aesthetic choice* as to which of the various approaches is used<sup>4</sup>. Nonetheless, given the vagaries of misinterpretation that can occur in application, it appears that the potentials-based approach considered herein, being one that follows from a well-defined mathematical structure implicit in the theory of partial differential equations, has much to offer and therefore comes well-recommended as a canonical procedure of choice.

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<sup>4</sup> As to whether the various options might be differentiated on the basis of tests by use of other than electromagnetic means, to be precise the answer is yes. Due to the difference in energy density *distributions* of the various energy density/power flux alternatives, discrimination among them could, at least in principle, be discerned on the basis of gravitational energy density measurements.